Example 1.2.6 *For a* ***power*** $x^n, n \in \mathbb{N}$ *we have, by the Product Rule,*

$$\lim_{x \to a} x^n = \left(\lim_{x \to a} x \right)^n = a^n.$$

For any ***polynomial*** $p(x) = c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0$*, we have, by the Sum Rule,*

$$\lim_{x \to a} p(x) = \sum_{i=0}^{n} c_i \lim_{x \to a} x^i = \sum_{i=0}^{n} c_i a^i = p(a)\,.$$

This says: The **limit** of a polynomial at a point is the **value** of the polynomial at that point.

Example 1.2.7 *A* ***rational*** *function is the quotient of polynomials, so* $r(x)$ *is a rational function if, and only if, it can be written as* $p(x)/q(x)$ *for some polynomials* $p(x)$ *and* $q(x)$*. Then*

$$\lim_{x \to a} r(x) = \lim_{x \to a} \frac{p(x)}{q(x)} = \frac{\lim_{x \to a} p(x)}{\lim_{x \to a} q(x)}$$

by the quotient rule, provided $\lim_{x \to a} q(x) = q(a)$ *is non-zero.*

Thus, since the limits of these polynomials equal their values at the limit point,

$$\lim_{x \to a} r(x) = \frac{p(a)}{q(a)} = r(a)\,.$$

This says The **limit** of a rational function at a point is the **value** of the rational function at that point, *provided that value is defined*.

Example 1.2.8 *As particular examples we deduce*

$$\lim_{x \to 2} (x + 3) = 2 + 3 = 5,$$

and

$$\lim_{x \to 2} \left(x^2 + 2x + 2 \right) = 4 + 4 + 2 = 10.$$

Then, since $5 \neq 0$*, we can use the Quotient Rule to deduce,*

$$\lim_{x \to 2} \frac{x^2 + 2x + 2}{x + 3} = \frac{\lim_{x \to 2} (x^2 + 2x + 2)}{\lim_{x \to 2} (x + 3)} = \frac{10}{5} = 2,$$

as has been proved earlier by verifying the $\varepsilon - \delta$ *definition.*

Note 1 We can **not** use the Quotient Rule to calculate

$$\lim_{x\to 1}\frac{x^2-1}{x^2-x}.$$

This is because $\lim_{x\to 1} q(x) = \lim_{x\to 1}(x^2-x) = 0$, and so the necessary conditions of the Theorem 1.2.5 are **not** satisfied.

Note 2 The Rules for Limits also hold if $x \to a$ is replaced by either of the one-sided limits $x \to a+$, $x \to a-$ or limits at infinity $x \to +\infty$ or $x \to -\infty$. It would be useful for the student to modify the proof I have given to show that it holds in these cases.

Recalling $\lim_{x\to+\infty} 1/x = 0$, proved by verifying the definition, means that by the Product Rule for limits at infinity

$$\lim_{x\to+\infty}\frac{1}{x^n} = \left(\lim_{x\to+\infty}\frac{1}{x}\right)^n = 0,$$

for all $n \geq 1$.

This simple result has applications as in the following.

Example 1.2.9

$$\lim_{x\to+\infty}\frac{4x^2+2}{2x^2+4x} = 2.$$

Solution Divide top and bottom by the largest power of x, namely x^2 to get

$$\lim_{x\to+\infty}\frac{4x^2+2}{2x^2+4x} = \lim_{x\to+\infty}\frac{4+2/x^2}{2+4/x} = \frac{\lim_{x\to+\infty}(4+2/x^2)}{\lim_{x\to+\infty}(2+4/x)},$$

by the Quotient Rule, *allowable* since both limit top and bottom both exist and the bottom one is non-zero. Thus

$$\lim_{x\to+\infty}\frac{4x^2+2}{2x^2+4x} = \frac{\lim_{x\to+\infty}(4+2/x^2)}{\lim_{x\to+\infty}(2+4/x)} = \frac{4}{2} = 2.$$

■

Note We cannot say

$$\lim_{x\to+\infty}\frac{4x^2+2}{2x^2+4x} = \frac{\lim_{x\to+\infty}(4x^2+2)}{\lim_{x\to+\infty}(2x^2+4x)},$$

because neither of the limits on the right hand side exist.

Theorem 1.2.10 ***Sandwich Rule:***

Suppose that f, g and h are three functions such that

$$h(x) \leq f(x) \leq g(x)$$

for all x in some deleted neighbourhood of a.

If $\lim_{x\to a} h(x) = L$ and $\lim_{x\to a} g(x) = L$ then $\lim_{x\to a} f(x) = L$.

Proof By the assumption in the Theorem there exists $\delta_0 > 0$ such that if $0 < |x - a| < \delta_0$ then $h(x) \leq f(x) \leq g(x)$.

Let $\varepsilon > 0$ be given.

From the definition of $\lim_{x\to a} h(x) = L$ there exists $\delta_1 > 0$ such that

$$\begin{aligned} 0 < |x-a| < \delta_1 \quad &\Longrightarrow \quad |h(x) - L| < \varepsilon \\ &\Longrightarrow \quad L - \varepsilon < h(x) < L + \varepsilon \\ &\Longrightarrow \quad L - \varepsilon < h(x)\,. \end{aligned}$$

From the definition of $\lim_{x\to a} g(x) = L$ there exists $\delta_2 > 0$ such that

$$\begin{aligned} 0 < |x-a| < \delta_2 \quad &\Longrightarrow \quad |g(x) - L| < \varepsilon \\ &\Longrightarrow \quad L - \varepsilon < g(x) < L + \varepsilon \\ &\Longrightarrow \quad g(x) < L + \varepsilon. \end{aligned}$$

Let $\delta = \min(\delta_0, \delta_1, \delta_2) > 0$ and assume $0 < |x - a| < \delta$. For such x we have all of $h(x) \leq f(x) \leq g(x)$, $L - \varepsilon < h(x)$ and $g(x) < L + \varepsilon$. Combine as in

$$L - \varepsilon < h(x) \leq f(x) \leq g(x) < L + \varepsilon,$$

i.e. $|f(x) - L| < \varepsilon$.

Thus we have verified the definition of $\lim_{x\to a} f(x) = L$. ■

Note The Sandwich rule also holds if $x \to a$ is replaced throughout by $x \to a^+$ or a^-, or $x \to +\infty$ or $x \to -\infty$.

Example 1.2.11 *Let*

$$f(x) = (x+1)^2 \sin(10(x+1)) - 1.$$

Find $\lim_{x\to -1} f(x)$.

Solution Start from the simple fact that $-1 \le \sin\theta \le 1$ for all θ. Hence

$$-1 \le \sin(10(x+1)) \le 1.$$

Thus

$$-(x+1)^2 - 1 \le (x+1)^2 \sin(10(x+1)) - 1 \le (x+1)^2 - 1.$$

By the product and sum rules for limits we have

$$\lim_{x\to-1}\left(-(x+1)^2 - 1\right) = -\left(\lim_{x\to-1} x+1\right)^2 - 1 = -1$$

and

$$\lim_{x\to-1}\left((x+1)^2 - 1\right) = -1.$$

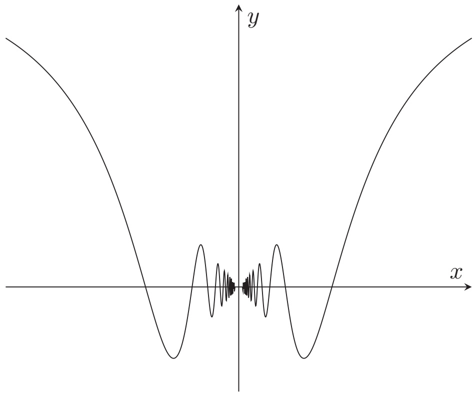
So, by the Sandwich rule,

$$\lim_{x\to-1}\left((x+1)^2 \sin(10(x+1)) - 1\right) = -1.$$

■

Example 1.2.12 *Prove that*

$$\lim_{\theta \to 0} \theta \sin\left(\frac{\pi}{\theta}\right) = 0.$$



Solution Start from the fact that, for *any* $\alpha \in \mathbb{R}$ we have

$$-|\alpha| \leq \alpha \leq |\alpha|.$$

In fact more is true, either $\alpha = |\alpha|$ or $\alpha = -|\alpha|$ but the inequality is all we require. Apply this with $\alpha = \theta \sin(\pi/\theta), \theta \neq 0$, to get

$$-\left|\theta \sin\left(\frac{\pi}{\theta}\right)\right| \leq \theta \sin\left(\frac{\pi}{\theta}\right) \leq \left|\theta \sin\left(\frac{\pi}{\theta}\right)\right|.$$

Then since $|\sin(\pi/\theta)| \leq 1$ we deduce

$$-|\theta| \leq \theta \sin\left(\frac{\pi}{\theta}\right) \leq |\theta|,$$

for $\theta \neq 0$. Finish off quoting the Sandwich Rule along with $\lim_{\theta \to 0} |\theta| = 0$. ■

Perhaps this figure will show what is happening: